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Purely Absolutely Continuous Spectrum for Almost Mathieu Operators

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Using a recent result of Sinai, we prove that the almost Mathieu operators acting on $l^2(\mathbb{Z})$, $(H_{\alpha,\lambda}\Psi)(n) = \Psi(n+1) + \Psi(n-1) + \lambda\cos(\omega n + \alpha)\,\Psi(n)$, have a purely absolutely continuous spectrum for almost all α provided that ω is a good irrational and λ is sufficiently small. Furthermore, the generalized eigenfunctions are quasiperiodic.

KEY WORDS: Schrödinger equation; quasiperiodic potential; Harper's equation; localization.

1. INTRODUCTION

In this paper we study the family of self-adjoint operators $H_{\alpha,\lambda}$ acting in $l^2(\mathbb{Z})$,

$$(H_{\alpha,\lambda}\Psi)(n) = \Psi(n+1) + \Psi(n-1) + \lambda\cos(\omega n + \alpha)\,\Psi(n) \tag{1}$$

where ω is a typical irrational number (see below Theorem 1). This operator is called the almost (or discrete) Mathieu operator and the associated eigenvalue problem is well known among physicists as the Harper equation.

This operator has been extensively studied since the pionering work by Azbel, André, and Aubry[(1)] and we recall the main features of this model of interest here. The spectrum (as a set) of this operator is a Cantor set at least for generic values of ω and λ.[(2)] Thus, the spectrum contains infinitely many gaps which can be labeled by the integrated density of states k: in the

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gaps k is constant and takes its value in the set $n\omega \bmod 1$, where n is an integer. The spectral measure does not contain any absolutely continuous component as soon as $\lambda > 2$,[3] and there is no eigenvalue for $\lambda < 2$.[4] Furthermore, using KAM techniques, it has been proved that for λ sufficiently small (and for typical ω) there exists an absolutely continuous part in the spectrum[5] and conversely there exist eigenvalues for large λ.[6] More recently, Sinai[7] has shown that it is possible to overcome the usual limitations of the KAM techniques and to tackle exactly the problem of resonances to prove that the spectral measure is only pure point for λ sufficiently large and for almost all α. An analogous result has been proved independently by Fröhlich *et al.*[8] In this note, we deduce from the above result that the spectral measure is purely absolutely continuous for small λ and for almost all α.

Theorem 1. For almost all ω (see remark at the end of this note) and for λ sufficiently small, the spectral measure of $H_{\alpha,\lambda}$ is purely absolutely continuous for almost all α. Furthermore, on this set of ω, λ, α, spectrally almost surely, the generalized eigenfunctions are quasiperiodic in the l^2 sense.

This theorem corresponds exactly to that of Sinai for large λ. It suggests that, as for large λ, the usual singularities of the KAM perturbation theory could be overcome to give rise to a complete perturbative expansion in terms of quasiperiodic eigenfunctions.

2. SINAI RESULT AND AUBRY DUALITY

Let us introduce the Hilbert space $\mathscr{H} = L^2(T^1) \otimes l^2(\mathbb{Z})$, which consists of functions $\Psi(\alpha, n)$ satisfying

$$\sum_n \int d\alpha\ \Psi(\alpha, n)^2 < +\infty$$

The Hilbert space $\mathscr{H}$ is decomposed as $\int \mathscr{H}_\alpha\, d\alpha$ and we can define the operator H_λ as

$$H_\lambda = \int H_{\alpha,\lambda}\, d\alpha \tag{2}$$

Then the following operator U from $\mathscr{H}$ into itself

$$(U\Psi)(\beta, p) = \int d\alpha \sum_n e^{i\alpha p + in(\beta + p\omega)} \Psi(\alpha, n) \tag{3}$$

is unitary [as can be checked by using the density in $\mathscr{H}$ of the linear combinations of the functions $\exp(i\alpha p)\,\Psi(n)$ for arbitrary Ψ in $l^2(\mathbb{Z})$ and p in $\mathbb{Z}$]. The inverse operator U^{-1} is given by

$$(U^{-1}\Psi)(\alpha, n) = \int d\beta \sum_p e^{-i\alpha p - in(\beta + p\omega)} \Psi(\beta, p) \tag{4}$$

Through the unitary operator U, H_λ is conjugated to the operator $(\lambda/2)\, H_{4/\lambda}$. We now recall Sinai's result:

Theorem (Sinai). For almost every ω and for λ sufficiently large, there exist measurable multivalued functions $E^k(\alpha)$ and $\Psi^k(\alpha)$ (with k almost surely finite) such that:

(i) $E^k(\alpha)$ is an eigenvalue of $H_{\alpha,\lambda}$ and $\Psi^k(\alpha)$ the corresponding eigenvector.

(ii) $\{E^k_m(\alpha) = E^k(\alpha + m\omega),\ \Psi^k_m(\alpha, n) = \Psi^k(\alpha + m\omega, n - m)\}$ is a complete set of eigenvalues and eigenvectors of $H_{\alpha,\lambda}$ for almost all α.

We show that this special covariant set of eigenvectors induces a natural spectral decomposition of $H_{4/\lambda}$ through the unitary conjugation defined by (3).

In the L^2 sense, any vector $\Psi(\alpha, n)$ in $\mathscr{H}$ can be written as

$$\Psi = \sum_{k,m} \Psi^k_m (\Psi^k_m, \Psi)_\alpha \tag{5}$$

where $(\cdot, \cdot)_\alpha$ is the inner product in $\mathscr{H}_\alpha$. Thus, by duality, we get

$$\begin{aligned} U\Psi &= U \sum_{k,m} \Psi^k_m (\Psi^k_m, \Psi)_\alpha \\ &= \sum_{k,m} U\Psi^k_m (\Psi^k_m, \Psi)_\alpha \end{aligned} \tag{6}$$

since U is unitary and Ψ^k_m is a sequence of orthonormal vectors. We have

$$\begin{aligned} U\Psi(\beta, p) &= \sum_{k,m} \int d\alpha \sum_n e^{i\alpha p + in(\beta + p\omega)} \Psi^k(\alpha - m\omega, n - m)(\Psi^k_m, \Psi)_\alpha \\ &= \sum_{k,m} \int d\alpha \sum_n e^{i(\alpha - m\omega)p + in(\beta + p\omega)} \Psi^k(\alpha, n - m) \\ &\quad \times \sum_q \Psi^k(\alpha, q - m)(\Psi)(\alpha - m\omega, q) \\ &= \sum_{k,m} \int d\alpha\, \Phi^k(\alpha, \beta, p)\, e^{im\beta} \sum_q \Psi^k(\alpha, q)\, \Psi(\alpha - m\omega, q + m) \end{aligned} \tag{7}$$

where

$$\Phi^k(\alpha, \beta, p) = \sum_n e^{i\alpha p + in(\beta + p\omega)} \Psi^k(\alpha, n)$$

For any p, $\Phi^k(\alpha, \beta, p)$ is square integrable with respect to α and β. Thus, if $\Psi(\alpha, m)$ decays sufficiently rapidly in m, then $\sum_q \Psi^k(\alpha, q)$ $\Psi(\alpha - m\omega, q + m)$ decays rapidly in m and in the L^2 sense we have for a dense set of Ψ in $\mathscr{H}$:

$$\begin{aligned}
U\Psi(\beta, p) &= \sum_k \int d\alpha\, \Phi^k(\alpha, \beta, p) \sum_m \sum_q \Psi^k(\alpha, q)\, e^{im\beta} \Psi(\alpha - m\omega, q + m) \\
&= \sum_k \int d\alpha\, \Phi^k(\alpha, \beta, p) \sum_m \sum_q \Psi^k(\alpha, q)\, e^{im\beta} \\
&\quad \times \int d\gamma \sum_j e^{-i\alpha j - iq(\gamma + j\omega) - im\gamma} (U\Psi)(\gamma, j) \\
&= \sum_k \int d\alpha\, \Phi^k(\alpha, \beta, p) \sum_q \Psi^k(\alpha, q) \sum_m e^{im\beta} \\
&\quad \times \int d\gamma \sum_j e^{-i\alpha j - iq(\gamma + j\omega) - im\gamma} (U\Psi)(\gamma, j) \\
&= \sum_k \int d\alpha\, \Phi^k(\alpha, \beta, p) \sum_q \Psi^k(\alpha, q) \sum_j e^{-i\alpha j - iq(\beta + j\omega)} (U\Psi)(\beta, j) \qquad (8)
\end{aligned}$$

the last equality holds since if $\Psi(\alpha, m)$ decays sufficiently rapidly in m, then $(U\Psi)(\gamma, j)$ is regular in γ. Finally, we get for a dense set of Ψ in $\mathscr{H}$

$$\Psi(\beta, p) = \sum_k \int d\alpha\, \Phi^k(\alpha, \beta, p) \sum_q \Phi^k(\alpha, \beta, q)^* \, \Psi(\beta, q) \qquad (9)$$

where the * holds for the complex conjugacy.

Furthermore, one easily checks that $\Phi^k(\alpha, \beta, p)$ satisfies formally

$$(\lambda/2)\, H_{\beta, 4/\lambda} \Phi^k(\alpha, \beta, \cdot) = E^k(\alpha)\, \Phi^k(\alpha, \beta, \cdot) \qquad (10)$$

Thus, for almost every α and β, (10) holds and (9) looks like the spectral decomposition of $H_\lambda^{(9)}$ in terms of the quasiperiodic (in the l^2 sense) functions $\Phi^k(\alpha, \beta, p)$. The conclusion now requires that the measure on E induced by the measure $d\alpha$ is absolutely continuous with respect to the Lebesgue measure. This relies on the following lemma, which is a consequence of the Sinai proof.

Lemma 1. Let $F^k(E)$ be the distribution function of values of $E^k(\alpha)$:

$$F^k(E) = \text{mes}\{\alpha: E^k(\alpha) \leqslant E\}$$

Then for any $s \in [1, 2)$

$$\sum_k \int |dF^k(E)/dE|^s \, dE < \infty$$

Thus (9) can be written as

$$\Psi(\beta, p) = \sum_k \int |d\alpha/dE^k(\alpha)| \, dE \, \Phi^k(\alpha, \beta, p) \sum_q \Phi^k(\alpha, \beta, q)^* \, \Psi(\beta, q) \quad (11)$$

Notice that α may be a multivalued E^k, thus, the integral has to be computed by summing over all branches of α: the sum over all branches of $|d\alpha/dE^k(\alpha)| \, dE$ provides a measure equal to $|dF^k(E)/dE| \, dE$. Thus, Eq. (11) is exactly the spectral decomposition of the operator $H_{\beta, 4/\lambda}$ in terms of the Lebesgue measure dE and of the quasiperiodic functions $\Phi^k(\alpha, \beta, \cdot)$. This ends the proof of Theorem 1 and we now prove Lemma 1.

Proof of Lemma 1. The proof of $L^s(\mathbb{R})$ regularity of the multivalued function $\alpha(E)$ inverse to $E(\alpha) = \{E^k(\alpha)\}$ is based on a very detailed analytical description of $\{E^k(\alpha)\}$ given in ref. 7. Using the inductive procedure of that paper, we can represent the countable family $\{E^k(\alpha)\}$ in the following way:

$$\{E^k(\alpha)\} = \bigcup_{t=0}^{\infty} \{\Lambda^{t,m}(\alpha), 1 \leqslant m \leqslant M(t)\}$$

where $M(t)$ has a polynomial upper bound, $M(t) < P(t)$. The degree of P depends on ω. The domain of definition of each function $\Lambda^{t,m}(\alpha)$ is a nowhere dense Cantor set. This set is constructed rather implicitly, but it follows from inductive assumptions in ref. 7 that this set is contained in an interval of length less then $\exp(-Ct)$. The constant C, as well as several other constants throughout this proof, depends on λ and is bounded away from 0 for all sufficiently large λ. Furthermore, $\Lambda^{t,m}(\alpha)$ can be written as

$$\Lambda^{t,m}(\alpha) = \tilde{\Lambda}^{t,m}(\alpha)[1 - \chi_{t,m}(\alpha)]$$

where $\chi_{t,m}(\alpha)$ is the characteristic function of a countable union of segments on which $\Lambda^{t,m}$ is not defined. As to $\tilde{\Lambda}^{t,m}$, it has exactly one critical point (maximum or minimum) and its second derivative $\Lambda^{t,m}$ at the critical point is greater than $\exp(bt)$ by absolute value, $b>0$. Respectively, the density of the distribution of values for $\tilde{\Lambda}^{t,m}$ (hence for $\Lambda^{t,m}$) has a unique singularity of $L^s(\mathbb{R})$ type, $(A^{t,m})^{-1}\,|E-E^{t,m}|^{-1/2}$. Summing over all $M(t)$ branches $\Lambda^{t,m}$, we get a multivalued function Λ^t for which the density of the distribution of values is less than $P(t)\exp(-C_1 t)<\exp(-C_2 t)$. Summation over all $t\geqslant 0$ completes the proof.

Remark. The frequency ω was assumed in ref. 7 to satisfy the following Diophantine condition: the coefficients l_n of the continued fraction $\omega = [l_1, l_2,..., l_n,...]$ grow not faster than const $\cdot\, n^2$. So, rigorously speaking, the above lemma is proved for such typical ω, since we just refer to the inductive assumptions of ref. 7. However, the condition $l_n \leqslant \text{const}\cdot n^2$ is certainly not optimum, and we suggest that any rate slower than exponential can be treated in a similar way.

REFERENCES

1. M. Ya. Azbel', *Sov. Phys. – JETP* **19**:634 (1964); S. Aubry and G. André, *Ann. Israel Phys. Soc.* **3**:133 (1980).
2. J. Bellissard and B. Simon, *J. Funct. Anal.* **48**:408 (1982).
3. J. Avron and B. Simon, *Duke Math. J.* **50**:369 (1983); L. A. Pastur and A. L. Figotin, *J. Math. Phys.* **25**:774 (1984).
4. F. Delyon, *J. Phys. A* **20**:L21 (1987).
5. E. I. Dinaburg and Ya. Sinai, *Funct. Anal. Appl.* **9**:279 (1975); E. D. Belokolos, *Theor. Math. Phys.* **25**:276 (1975); Ya. Sinai, *Funct. Anal. Appl.* **19**:42 (1985); R. A. Johnson and J. Moser, *Commun. Math. Phys.* **84**:403 (1982); H. Russmann, *Ann. N. Y. Acad. Sci.* **357**:90 (1980).
6. J. Bellissard, R. Lima, and D. Testard, *Commun. Math. Phys.* **88**:207 (1983).
7. Ya. Sinai, *J. Stat. Phys.* **46**(5/6):861 (1987).
8. J. Fröhlich, T. Spencer, and P. Wittwer, Localization for a class of one dimensional quasi-periodic Schrödinger operators, Preprint.
9. I. M. Gelfand and N. Ya. Vilenkin, *Generalized Functions*, Vol. 4 (Academic Press, 1964).

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